Non-Gaussian Optimization

Absolute return with positive asymmetry preferences

Introduction

In this Quant Corner, we present a simplified allocation process taking into account preference for positive asymmetry. This allocation process is composed of 3 main models: the performance, risk and optimization models.

First order performance signals can be the output of a quantitative model (momentum, value, mean-reversion, …) or the result of a fundamental analysis.

In this paper, the risk and allocation models are purely quantitative.

We present the incremental approach used to structure this quantitative allocation:

1. improving the reactivity of the second order model,
2. adding non-Gaussian constraints in the allocation model to take into account preferences for positive asymmetry.

Best practices to improve as far as possible the second order risk model

The goal of this part is to design second-order reactive and robust risk-driven decision models to detect soon enough dangerous positions.

Short windows and adapted filtering process

The first way to obtain reactivity and capture recent market changes consists in reducing drastically the size of the estimation windows for the computation of covariance matrices. Typically, our models went through the crisis with 60 days windows!
However, this cannot be done without caution: it is only relevant if combined with an adapted filtering process that enables to deal with lack of data and sampling risk introduced by the shortened windows.

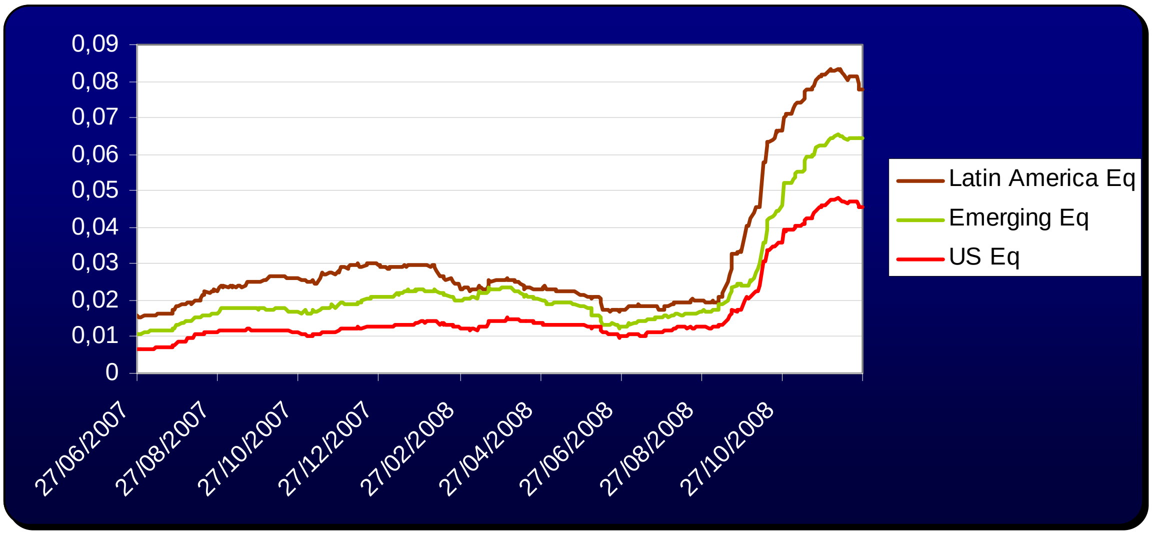
The proposed filtering process (ref Quant Corner "Risk Filtering") has its justification in two fields of the mathematics: the random matrix theory and the robust optimization theory.

Random matrix theory (ref [Laloux & al 1999]) proves that the main drawback of empirical covariance matrices (especially when estimated with a few data) is the under-estimation of the smallest contributions to risk (the smallest eigenvalues).
Furthermore, the robust control theory ([El Ghaoui, Oustry & al 1998], [Ben Tal & al 1998], [Ben-Tal & al 2009]) explains that the use of such bad estimated covariance matrices as input of optimization problems certainly leads to irrelevant and non robust optimal portfolios.
The filtering step consists in restoring good properties (from an optimization point of view, that means: a good conditioning number) while preserving initial information of the empirical covariance matrix. This problem can be formulated as a Semi Definite Program.

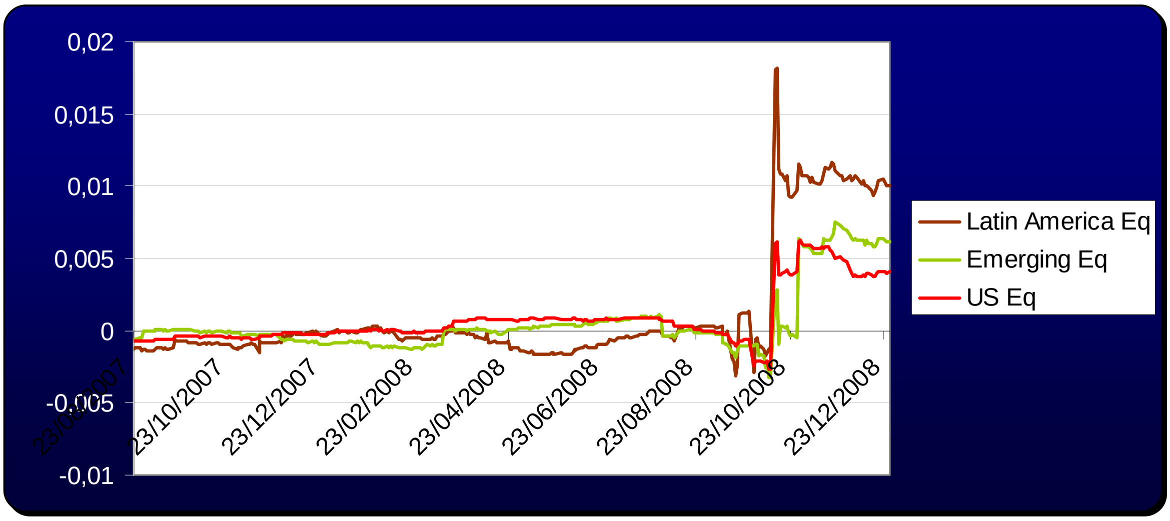
Remark: Other ways to boost reactivity exist that could be the subject of another paper (in particular: adding implied information that gave very interesting results during the 2008 crisis).

Adding non-Gaussian constraints into the model

Observations and questions



Evolution of the volatility of Equity indices before and during the crisis



Evolution of the third moment for Equity indices before and during the crisis

The observation of volatilities and third moment dynamics during 2007-2009 period leads to one remark: when estimated with short windows, these indicators seem to be rather good sentinels to detect market changes.
From this observation, the question is: **How can we take a systematic benefit of this ex ante information?**
The challenge here is: **are we able to take into account numerical estimations of mathematical objects such as co-skewness coefficients -** that are known to be very unstable **- to improve ex-post performances?**

An incremental approach

The first step consists in computing the best second-order portfolio: ω_0. Then, the model aims at capturing the relevant signals to improve skewness around the reference portfolio. Once these directions are determined, the deviation from the reference portfolio are accepted as far as the first and second order indicators are not too much deteriorated. The optimization problem is then formulated as explained below.

A non-convex problem

Once reformulated, the problem consists in minimizing portfolio volatility penalized by a third order moment term with respect to a return constraint, the investment linear constraints and a proximity to the reference portfolio constraint.

The optimization problem is then formulated as :

$$\min_{\omega} \ \omega^t \Gamma \omega - \kappa H(\omega)$$
$$\begin{cases} \rho' \omega \geq \mu \\ 1' \omega = 1 \\ \omega \in \Delta \\ \omega \in V(\omega_0) \end{cases}$$

Where:

- ω is a portfolio represented by a n-vector
- Γ is the covariance nxn-matrix of returns of the assets constituting the portfolio
- ρ is the n-vector of expected returns of the assets constituting the portfolio
- μ is the ex-ante performance target
- Δ is the set of linear constraints that the portfolio has to respect
- $V(\omega_0)$ is a neighbourhood of the portfolio ω_0.

The challenges of such a problem are:

- the non convexity of the problem ;
- the ex-ante estimation of the third-order moment;
- the calibration of 3 parameters : the size of the estimation for the third order moment, the coefficient of asymmetry appetence k and the distance to the reference portfolio.

The objective function of the problem is a third-order polynomial function of weights which is not convex (ref [Campbell & al 2004], [Jondeau & al 2006]). In order to preserve stability properties as well as computational cost known in advance, these problems can be approximate using a specific convex relaxation: *lift-and-project procedures* proposed by [Lovász & al 1991] resulting in Semidefinite Program [Laurent 2008].

The process consists in transforming the third-order term of the objective function into a second order term by changing the variable space: from the space of n-vectors to the space of nxn matrices.

To do so, we introduce:

- W a $n \times n$ matrix such as $W \succeq \omega\omega^T$
  $\tilde{W} = mat2vec(W): \ p \ vector \ where \ \ p = n(n+1)/2$

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- $\Omega = \begin{bmatrix} \tilde{W} \\ \omega \end{bmatrix} : N\ vector\ where\ N = p + n$
- $H\ /\ H_{ijk} = \frac{1}{m}\sum_{t=1}^{m}\left(\rho_i^t - \bar{\rho}_i\right)\left(\rho_j^t - \bar{\rho}_j\right)\left(\rho_k^t - \bar{\rho}_k\right)$

  $H : n \times n \times n\ tensor$

- $\tilde{H} = \begin{bmatrix} 0 & tens2mat(H) \\ tens2mat(H)^T & 0 \end{bmatrix}$

  $tens2mat(H) : p \times n\ matrix$

  $\tilde{H} : N \times N\ matrix$

- Z a $N \times N$ matrix such as $Z \succeq \Omega\Omega^T$

Then, we get a second order objective function of the new variable (ω, W). A similar transformation enables to get a linear objective function of another bigger variable that can be view as a big matrix (ω, W, Z).

$$\begin{cases} \min\limits_{W \geq 0, Z \geq 0, \omega} \left\langle \Gamma, W \right\rangle - \kappa \left\langle \tilde{H}, Z \right\rangle \\ s.t \\ \qquad \rho' \omega \geq \mu \\ \qquad 1'\omega = 1 \\ \qquad \omega \in \Delta \\ \qquad \omega \in V(\omega_0) \\ \qquad W \succeq \omega\omega^T \\ \qquad Z \succeq \Omega\Omega^T \\ \qquad tr(Z) \leq N \end{cases}$$

$$\begin{bmatrix} W & \omega \\ \omega^T & 1 \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} Z & \Omega \\ \Omega^T & 1 \end{bmatrix} \succeq 0$$

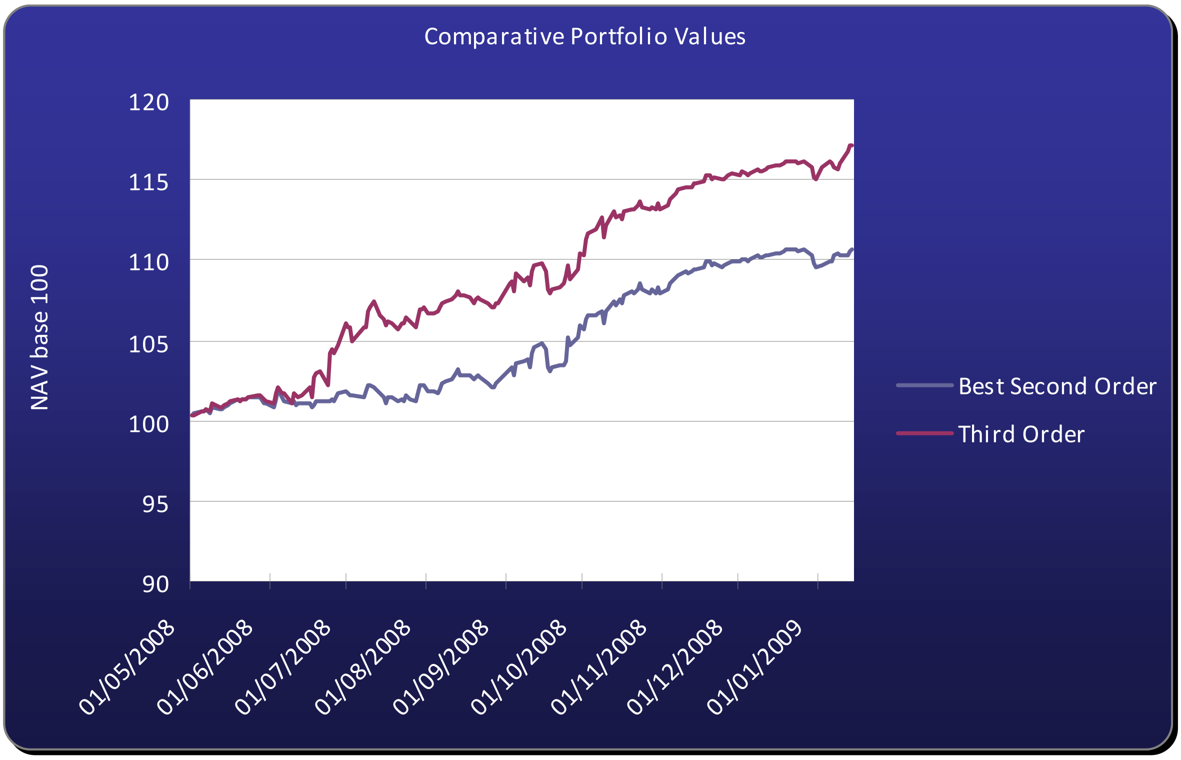
Note that thanks to the Schur lemma, the 2 colored constraints can be reformulated as SDP constraints.

Finally the problem can be rewritten as a semi definite program: $\begin{cases} \max & \left\langle C, X \right\rangle \\ & X \succeq 0 \\ & l_i \leq \left\langle X, A_i \right\rangle \leq u_i \quad A_i \in S_n \end{cases}$

Where $X = (\omega, W, Z)$ *is* the new variable. The very large size of *X* leads us to use bundle methods to solve this problem using the fact that the dual of the SDP problem is in fact here a maximum eigenvalue problem [Helmberg & Oustry 2000].

Some results

In the graphs bellow are illustrated the gain (in term of performance) when using such algorithms for optimizing a quantitative Global Macro strategy during the recent crisis.



Compared NAV during the crisis period

Note on the realized CVaR (Conditional Value-at-Risk)

In practise the tuning of the attractiveness for positive third-order moment enables us to control efficiently the realized CVaR.

Alternatively the convex non-smooth optimizations techniques (bundle methods [Hiriart-Urruty & Lemaréchal 1993]) are used in NORM Asset Management® to solve CVaR optimization problems.

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